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On the unique solvability of the equation describing unsteady motion of a stratified fluid.

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§1. Introduction.

The Navier-Stokes equation for incompressible (homogeneous) viscous fluid motion is derived from the following conditions;

i) conservation of mass,

ii) equation of motion,

iii) constancy of mass with respect to the time or the space variables.

If fluid is inhomogeneous, then the condition iii) is not equivalent to the incompressibility. When we deal with inhomogeneous fluid, we must replace the condition iii) with

iii)' the Lagrange derivative of the mass ρ is zero.

This condition means that ρ is constant along the stream of particles. The conditions i), ii) and iii)' lead us to the following system:

$$\frac{\partial \rho}{\partial t} + u\cdot\nabla\rho = 0 \qquad (0 < t, x \in \Omega), \tag{1.1}$$

$$\rho\{\frac{\partial u}{\partial t} + (u\cdot\nabla)u\} = \Delta u - \nabla p \qquad (0 < t, x \in \Omega), \tag{1.2}$$

$$\mathrm{div}\ u = 0 \qquad (0 < t, x \in \Omega). \tag{1.3}$$

Here the unknowns ρ , u and p mean the mass, the velocity and the pressure, respectively. Ω is a domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ occupied by the fluid. In this note we solve the system (1.1), (1.2) and (1.3) under the initial-boundary condition below;

(1.4) $$u|_{\partial\Omega} = 0,$$

(1.5) $$u|_{t=0} = a(x) \ , \ \rho|_{t=0} = \rho_0(x).$$

The mathematical study for the initial value problem (1.1),$\cdots$,(1.5) was initiated by Kazhikhov [3]. There he proved existence of a weak solution of Hopf-type and also a classical solution. However, he could not show the uniqueness of the solution. Later Ladyzhenskaya and Solonnikov [4] proved unique existence in the framework of L^p-theory. However, they required that p is greater than the dimension of the domain Ω. On the other hand, Lions [5] proved existence of another weak solution without uniqueness even in the two-dimensional problem. Marsden [6] dealt with the case of inviscid inhomogeneous fluid, i.e., he considered the system in which the term Δu in (1.2) is dropped.

In this note we employ the L^2-theory and prove the unique existence local in time of the solution of (1.1),$\cdots$,(1.5). Furthermore we show the following global results:

I) In the case of the three-dimensional problem, the solution exists globally in time if the initial values a and ρ_0 are sufficiently small.

II) In the case of the two-dimensional problem, the solution always exists globally.

We only give statements of the results and outlines of the proofs. The details of the proof will be published elsewhere.

§2. Abstract formulation and main theorems.

Since the system (1.1),$\cdots$,(1.5) is an extension of the Navier-Stokes system, we solve the above system by the method employed in Fujita and Kato [1]. To this end we introduce some function spaces below. (n is the dimension of Ω).

$C_{0,\sigma}^{\infty}(\Omega) = \{ v = (v_1, v_2, \cdots, v_n) \in C_{0,\sigma}^{\infty}(\Omega)^n \; ; \; \mathrm{div}\, v = 0 \quad \text{in } \Omega \}$,

V = the closure of $C_{0,\sigma}^{\infty}(\Omega)$ in the usual Sobolev space $H_0^1(\Omega)^n$,

H = the closure of $C_{0,\sigma}^{\infty}(\Omega)$ in $L^2(\Omega)^n$ with L^2-norm $\|\ \ \|$.

$P \; ; \; L^2(\Omega)^n \longrightarrow H \quad \cdots$ the orthogonal projection,

A ; the Stokes operator in H, i.e., $D(A) = H^2(\Omega)^n \cap V$, $Au = -P\Delta u$ ($u \in D(A)$) (see Fujita and Kato [4]).

Hereafter we assume that Ω is a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with a smooth boundary. We recall that A is a positive definite self-adjoint operator in H.

We formulate (1.1),$\cdots$,(1.5) as a quasilinear evolution equation in $W^{1,\infty}(]0,T[\times \Omega) \times H$ as follows.

$$\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho = 0 \qquad \text{a.e. } (t,x) \in \,]0,T[\times \Omega ,$$

$$P\rho(t)\frac{du}{dt} + Au(t) + F_\rho u(t) = 0 \qquad (0 < t\,),$$

$$u(0) = a \quad , \quad \rho(0,x) = \rho_0(x).$$

Here we have put $\rho(t) = \rho(t,\cdot)$ and we have defined the non-linear operator F_ρ by $F_\rho w = P\{\rho(t)(w\cdot\nabla)w\}$. $W^{1,\infty}(]0,T[\times\Omega)$ is the usual Sobolev space (whose element are Lipschitz continuous). Our results are the following three theorems.

<u>THEOREM 2.1.</u> Assume that $a \in D(A^\eta)$ and $\rho_0 \in W^{1,\infty}(\Omega)$ ($\eta > n/4$) and that, for some positive numbers m and ℓ, ρ_0 satisfies $m < \rho_0(x) < \ell$ ($x \in \Omega$).

i) Then there exists a unique solution

$$u \in C([0,T_0];D(A^\eta))\cap C(]0,T_0];D(A))\cap C^1(]0,T_0];H) \quad \text{and}$$

$$p \in W^{1,\infty}(]0,T_0[\times\Omega)$$

so long as $m_0 < m$, $\ell < \ell_0$, $\|A^\eta a\| < k_1$, $\|\nabla\rho_0\|_{L^\infty} < k_2$. Here $T_0 = T_0(\Omega,m_0,\ell_0,k_1,k_2) > 0$ is a non-increasing function of positive variables m_0 , ℓ_0 , k_1 and k_2.

ii) Furthermore u and ρ satisfy the following inequalities.

(2.1) $$\|A^\alpha u(t)\| \leqq c_0\|A^\alpha a\| \qquad (\alpha = 5/8\ ,\ \eta\ ; 0 \leqq t \leqq T_0),$$

(2.2) $$\|\nabla\rho(t)\|_{L^\infty} \leqq c_0\|\nabla\rho_0\|_{L^\infty}\exp(c_0\|A^\eta a\|) \qquad (0 < t < T_0),$$

(2.3) $$\left\|\frac{\partial\rho}{\partial t}(t)\right\|_{L^\infty} \leqq c_0\|\nabla\rho_0\|_{L^\infty}\|A^\eta a\|\exp(c_0\|A^\eta a\|) \qquad (0<t<T_0).$$

Here c_0 is a positive constant depending only on Ω, m_0 and ℓ_0.

As for the global existence of the solution we present two theorems below, which are analogous to the results in the Navier-Stokes problem.

THEOREM 2.2. If Ω is a two-dimensional domain, then the solution always exists in $[0,\infty[$.

THEOREM 2.3. Consider the three-dimensional problem. Let $0 < m_0 < \ell_0 < \infty$ be given. Then there exists a positive constant $\varepsilon_1 = \varepsilon_1(\Omega, m_0, \ell_0)$ such that u and ρ exist in $[0,\infty[$ if $m_0 \leqq m$, $\ell \leqq \ell_0$, $\| A^\eta a \| \leqq \varepsilon_1$ and $\| \nabla \rho_0 \|_{L^\infty} \leqq \varepsilon_1$.

§3. Outline of the proofs of THEOREMS 2.1, 2.2 and 2.3.

PROOF OF THEOREM 2.1. We construct a solution as a limit of successive approximations $\{u_n, \rho_n\}$. First we put $u_1(t) = e^{-tA}a$ and $\rho_1(t,x) = \rho_0(x)$. When u_n and ρ_n are given, we define u_{n+1} and ρ_{n+1} by the following equation;

$$\frac{\partial \rho_{n+1}}{\partial t} + u_n \cdot \nabla \rho_{n+1} = 0 \qquad (0 < t < T, \ x \in \Omega), \tag{3.1}$$

$$\rho_{n+1}(0,x) = \rho_0(x), \tag{3.2}$$

$$P\rho_n(t)\frac{du_{n+1}}{dt} + Au_{n+1} = -F_{\rho_n} u_n(t) \qquad (0 < t < T), \tag{3.3}$$

$$u_{n+1}(0) = a. \tag{3.4}$$

Since these equations are linear ones, we can apply standard theories of linear equations of evolution. The initial value problem (3.1) and (3.2) are explicitly solved as

$$\rho_{n+1}(t,x) = \rho_0(\xi_{0,t}(x)),$$

where $\xi_{s,t}(x)$ is a characteristic curve, i.e.,

$$\frac{d}{ds}\xi_{s,t}(x) = u_n(s,\xi_{s,t}(x)), \qquad \xi_{t,t}(x) = x \ (\in \Omega).$$

We solve (3.3) and (3.4) by showing that $(P\rho_n(t))^{-1}A$ $(0 < t < T)$ generates an evolution operator $U(t,s)$ $(0 \leqq s \leqq t \leqq T)$ in H (see, e.g., Tanabe [8]).

PROOFS of THEOREMS 2.2 and 2.3. These theorems are derived from the following a priori estimates.

PROPOSITION 3.1. In the case of the two-dimensional problem, there exists, for any $T > 0$, a positive constant $L(T) = L(T,\Omega,\|A^\eta a\|, \|\nabla\rho_0\|_{L^\infty}, m, \ell)$ such that we have

(3.5) $$\|A^\eta u(t)\| \leqq L(T) \qquad (0 < t < T),$$

(3.6) $$\int_0^t \|\nabla u(s)\|_{L^\infty} ds \leqq L(T) \qquad (0 < t < T).$$

PROPOSITION 3.2. Consider the three-dimensional problem. Let $T > 0$, $0 < m_0 < \ell_0 < \infty$ be arbitrary numbers. Then there exist positive constants $\delta = \delta(\Omega)$, $c^* = c^*(\Omega,m_0,\ell_0)$ and $\varepsilon^* = \varepsilon^*(\Omega,m_0,\ell_0)$ satisfying the following property. If the inequalities $m_0 < m$, $\ell < \ell_0$,

$$\sup_{0<t<T} \left\| \frac{\partial\rho}{\partial t}(t) \right\|_{L^\infty} \leqq \varepsilon^* ,$$

$$\sup_{0<t<T} e^{\delta t} \| A^{5/8} u(t) \| \leqq \varepsilon^* \quad \text{and}$$

$\| A^\eta a \| \leqq 1$ are satisfied, then we have

$$\sup_{0<t<T} e^{\delta t} \| A^{5/8} u(t) \| \leqq c^* \| A^{5/8} a \| ,$$

$$\sup_{0<t<T} e^{\delta t} \| \nabla u(t) \|_{L^\infty} t^{1+\gamma-\eta} \leqq c^* \| A^\eta a \| ,$$

$$\sup_{0<t<T} \left\{ \left\| \frac{\partial\rho}{\partial t}(t) \right\|_{L^\infty} + \| \nabla\rho(t) \|_{L^\infty} \right\} \leqq c^* \| \nabla\rho_0 \|_{L^\infty} ,$$

where γ is an absolute constant in $]1/4, 3/8[$.

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